FRM Part 1 Formula Sheet 2026 edition

1 Foundations of Risk Management 2 Quantitative Analysis 3 Financial Markets and Products 4 Valuation and Risk Models

FOUNDATIONS OF RISK MANAGEMENT

FRM 1 The Building Blocks of Risk Management

RAROC

$$\text{RAROC} = \frac{\text{Reward}}{\text{Risk}};$$

in full: $\dfrac{\text{After-tax net risk-adjusted expected return}}{\text{Economic capital}}$

FRM 5 Modern Portfolio Theory and the CAPM

Beta

$$\beta_i = \frac{\text{Cov}(R_i, R_M)}{\sigma_M^2} = \rho_{i,M}\frac{\sigma_i}{\sigma_M},\ \beta_M = 1$$

Portfolio beta

$$\beta_P = \sum w_i\beta_i$$

CAPM (SML)

$$E(R_i) = r_f + \beta_i\,[E(R_M) - r_f]$$

slope of the SML $= E(R_M) - r_f$

CML

$$E(R_P) = r_f + \frac{E(R_M) - r_f}{\sigma_M}\sigma_P$$

slope of the CML $= [E(R_M) - r_f]/\sigma_M$

Sharpe performance index

$$\text{SPI} = \frac{E(R_P) - r_f}{\sigma_P}$$

Treynor performance index

$$\text{TPI} = \frac{E(R_P) - r_f}{\beta_P}$$

Jensen performance index

$$R_P - r_f = \alpha_P + \beta_P(R_M - r_f);$$
$$\alpha_P = E(R_P) - \{r_f + \beta_P[E(R_M) - r_f]\}$$

$\alpha_P = 0$ in equilibrium;

significantly positive is superior, negative is inferior

Tracking error

$$\text{TE} = \sigma(R_P - R_B)$$

Information ratio

$$\text{IR} = \frac{E(R_P - R_B)}{\sigma(R_P - R_B)}$$

active return over tracking error

Sortino ratio

$$\text{SR} = \frac{E(R_P) - \text{MAR}}{\sqrt{\text{MSD}_{\min}}};$$
$$\text{MSD}_{\min} = \frac{1}{N}\sum_{t=1}^{N}\min(0, R_{Pt} - \text{MAR})^2$$

> *Sharpe divides by total risk σ_P; Treynor divides by systematic risk β_P, so it suits a well-diversified portfolio.*

FRM 6 The APT and Multifactor Models

APT return model

$$R_i = E(R_i) + \beta_{i1}[I_1 - E(I_1)] + \cdots + \beta_{iK}[I_K - E(I_K)] + e_i$$

APT expected return

$$E(R_P) = E(R_Z) + \beta_{P1}[E(I_1) - E(R_Z)] + \cdots + \beta_{PK}[E(I_K) - E(R_Z)]$$

$E(R_Z)$ = zero-beta (risk-free) return; single factor: $K = 1$

Fama-French three-factor

$$E(R_P) = r_f + \alpha_P + \beta_{P,M}[E(R_M) - r_f] + \beta_{P,SMB}\,E(\text{SMB}) + \beta_{P,HML}\,E(\text{HML})$$

Portfolio factor beta

$$\beta_{P,k} = \sum w_i\beta_{i,k}$$

a hedge adds positions until $\beta_{P,k} = 0$ for each factor hedged

QUANTITATIVE ANALYSIS

QTA 1 Fundamentals of Probability

Complement

$$P(A^c) = 1 - P(A)$$

Addition rule

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Conditional probability

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

Total probability

$$P(B) = P(B \mid A_1)P(A_1) + \cdots + P(B \mid A_n)P(A_n)$$

A_i mutually exclusive and exhaustive

Total probability, two events

$$P(A) = P(A \mid B)P(B) + P(A \mid B^c)P(B^c)$$

Independence

$P(A \cap B) = P(A)P(B)$, i.e. $P(A \mid B) = P(A)$;

n events: $P(A_1 \cap \cdots \cap A_n) = P(A_1)\cdots P(A_n)$

Conditional independence

$$P(A \cap B \mid C) = P(A \mid C)\,P(B \mid C)$$

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Bayes' rule

$$P(B \mid A) = \frac{P(A \mid B)\,P(B)}{P(A)}$$

> *Mutually exclusive is not independent: two mutually exclusive events with positive probabilities are dependent.*

QTA 2 Random Variables

Bernoulli PMF

$$f(x) = p^x(1-p)^{1-x},\ x \in \{0,1\}$$

Expectation

$$E[X] = \sum x_i p_i; \quad E[f(X)] = \sum f(x_i)p_i$$

Properties of E

$$E[a + bX + cY] = a + bE[X] + cE[Y];$$
$$E[g(X)] \neq g(E[X]) \text{ in general}$$

Variance

$$\sigma^2 = E[(X-\mu)^2] = E[X^2] - (E[X])^2$$

Skewness

$$\text{Skew}(X) = \frac{E[(X-\mu)^3]}{\sigma^3}$$

numerator is the third central moment

Kurtosis

$$\text{Kurt}(X) = \frac{E[(X-\mu)^4]}{\sigma^4}; \quad \text{excess kurtosis} = \text{Kurt} - 3$$

Standardized variable

$$Z = \frac{X-\mu}{\sigma}$$

mean 0, variance 1

Linear transformation

$$Y = a + bX\text{: } E[Y] = a + bE[X],\ V[Y] = b^2V[X],$$
$$\sigma_Y = |b|\,\sigma_X$$

Higher moments of $Y = a + bX$

skewness unchanged if $b > 0$ (sign flips if $b < 0$);
kurtosis unchanged;
median and IQR move like the mean and SD

Median and IQR

median = 50% quantile; $\text{IQR} = q_{0.75} - q_{0.25}$

QTA 3 Common Univariate Random Variables

Binomial

$$P(X = x) = {}^nC_x\, p^x(1-p)^{n-x}$$

Poisson

$$P(X = x) = \frac{\lambda^x e^{-\lambda}}{x!}$$

over a period t, use λt

Uniform, interval probability

$$P(l < X < u) = \frac{\min(u,b) - \max(l,a)}{b-a}$$

Normal

$$P(X \le x) = \Phi\big((x-\mu)/\sigma\big); \quad \phi(z) = \frac{1}{\sqrt{2\pi}}e^{-z^2/2}$$

Normal facts

mean = median = mode; skewness 0; kurtosis 3;
$\Phi(-z) = 1 - \Phi(z)$

Normal probability ranges

68%: $\mu \pm \sigma$; 90%: $\mu \pm 1.645\sigma$; 95%: $\mu \pm 1.96\sigma$;
99%: $\mu \pm 2.58\sigma$

Lognormal

$$P(X \le x) = \Phi\big((\ln x - \mu)/\sigma\big)$$

positively skewed

Exponential

$$P(X \le x) = 1 - e^{-x/\beta},\ x \ge 0$$

Student's t kurtosis

$$\text{Kurt} = \frac{3(\nu-2)}{\nu-4}$$ for $\nu > 4$, always above 3

Constructions

$$\chi^2_\nu = \textstyle\sum_{i=1}^{\nu} Z_i^2; \quad t_\nu = \dfrac{Z}{\sqrt{W/\nu}}, W \sim \chi^2_\nu; \quad F_{\nu_1,\nu_2} = \dfrac{W_1/\nu_1}{W_2/\nu_2}$$

Mixture

$$f(x) = \sum w_i f_i(x); \quad E[X] = \sum w_i\mu_i;$$
$$V[X] = \sum w_i(\sigma_i^2 + \mu_i^2) - \left(\sum w_i\mu_i\right)^2$$

Distribution	Mean	Variance
Bernoulli	p	$p(1-p)$
Binomial	np	$np(1-p)$
Poisson	λ	λ
Uniform $[a,b]$	$(a+b)/2$	$(b-a)^2/12$
Normal	μ	σ^2
Standard normal	0	1
Lognormal	$e^{\mu+\sigma^2/2}$	$(e^{\sigma^2}-1)\,e^{2\mu+\sigma^2}$
Student's t_ν	0	$\nu/(\nu-2)$
F_{ν_1,ν_2}	$\nu_2/(\nu_2-2)$	not required
χ^2_ν	ν	2ν
Exponential	β	β^2
Beta(α,β)	$\alpha/(\alpha+\beta)$	not required

> *In the lognormal, μ and σ are the mean and SD of $\ln X$, not of X.*

QTA 4 Multivariate Random Variables

Marginal and conditional PMF

$$f_{X_1}(x_1) = \textstyle\sum_{x_2} f(x_1,x_2); \quad f(x_1 \mid X_2 = x_2) = \dfrac{f(x_1,x_2)}{f_{X_2}(x_2)}$$

Conditional expectation

$$E[X_1 \mid X_2 = x_2] = \sum x_1\, f(x_1 \mid X_2 = x_2)$$

Conditional variance

$$V[X_1 \mid X_2 = x_2] =$$
$$E[X_1^2 \mid X_2 = x_2] - (E[X_1 \mid X_2 = x_2])^2$$

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Covariance

$$\text{Cov}(X_1, X_2) = E[(X_1 - E[X_1])(X_2 - E[X_2])] = E[X_1X_2] - E[X_1]E[X_2]$$

Correlation

$$\text{Corr}(X_1, X_2) = \frac{\text{Cov}(X_1, X_2)}{\sigma_1\sigma_2}$$

so $\text{Cov} = \text{Corr} \cdot \sigma_1\sigma_2$

Linear transformations

$$\text{Cov}(a + bX_1, c + dX_2) = bd\,\text{Cov}(X_1, X_2);$$

$$\text{Corr}(aX_1, bX_2) = \frac{ab}{|a||b|}\text{Corr}(X_1, X_2)$$

unchanged if a, b have the same sign, flips sign otherwise

Covariance with a sum

$$\text{Cov}(X_1, X_2 + X_3) = \text{Cov}(X_1, X_2) + \text{Cov}(X_1, X_3)$$

Variance of a sum or difference

$$V[X_1 \pm X_2] = V[X_1] + V[X_2] \pm 2\,\text{Cov}(X_1, X_2)$$

Variance of a weighted sum

$$V[aX_1 + bX_2] = a^2V[X_1] + b^2V[X_2] + 2ab\,\text{Cov}(X_1, X_2)$$

Sum of n iid variables

$$E[\sum X_i] = n\mu; \quad V[\sum X_i] = n\sigma^2$$

QTA 5 Sample Moments

Sample mean

$$\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

Bias

$$\text{Bias}(\hat{\theta}) = E[\hat{\theta}] - \theta; \quad \text{Bias}(\hat{\mu}) = 0; \quad \text{Bias}(\hat{\sigma}^2) = -\sigma^2/n$$

Variance estimators

$$\hat{\sigma}^2 = \frac{1}{n}\sum(X_i - \hat{\mu})^2 \text{ (biased);}$$

$$s^2 = \frac{1}{n-1}\sum(X_i - \hat{\mu})^2 \text{ (unbiased)}$$

Standard error of the mean

$$\sigma/\sqrt{n}$$

Central limit theorem

$\hat{\mu} \approx N(\mu, \sigma^2/n)$ for large n; $\sqrt{n}\,(\hat{\mu} - \mu)/\sigma \to N(0, 1)$

Scaling the mean

$$\hat{\mu}_{\text{annual}} = 252\,\hat{\mu}_{\text{daily}} = 52\,\hat{\mu}_{\text{weekly}}$$

n periods: $n\mu$

Scaling volatility

$$\sigma_{\text{annual}} = \sqrt{252}\,\sigma_{\text{daily}} = \sqrt{52}\,\sigma_{\text{weekly}} = \sqrt{12}\,\sigma_{\text{monthly}}$$

n periods: $\sqrt{n}\sigma$

> *Square-root-of-time scaling assumes iid returns; it fails with autocorrelation.*

Sample skewness and kurtosis

$$\hat{S} = \frac{\frac{1}{n}\sum(X_i - \hat{\mu})^3}{\hat{\sigma}^3}; \quad \hat{\kappa} = \frac{\frac{1}{n}\sum(X_i - \hat{\mu})^4}{\hat{\sigma}^4}$$

Median

n odd: $x_{((n+1)/2)}$; n even: $\frac{x_{(n/2)} + x_{(n/2+1)}}{2}$

Sample covariance

$$\hat{\sigma}_{XY} = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \hat{\mu}_X)(Y_i - \hat{\mu}_Y)$$

Sample correlation

$$\hat{\rho}_{XY} = \frac{\hat{\sigma}_{XY}}{\hat{\sigma}_X\hat{\sigma}_Y}$$

QTA 6 Hypothesis Testing

Hypotheses

two-sided $H_0 : \mu = \mu_0$ vs $H_1 : \mu \neq \mu_0$;

one-sided $H_0 : \mu \leq \mu_0$ vs $H_1 : \mu > \mu_0$ (or the reverse)

Test statistic, mean

$T = \frac{\hat{\mu} - \mu_0}{\sqrt{\hat{\sigma}^2/n}} \sim N(0, 1)$ asymptotically;

with s^2 from a normal sample, $T \sim t_{n-1}$

Population	σ^2	$n \geq 30$	$n < 30$
Normal	Known	z	z
Non-normal	Known	z	NA
Normal	Unknown	t (or z)	t
Non-normal	Unknown	t (or z)	NA

Critical z	10%	5%	1%
One-sided	1.28	1.645	2.33
Two-sided	1.645	1.96	2.58

Decision	H_0 true	H_0 false
Do not reject H_0	Correct, $1 - \alpha$	Type II error, β
Reject H_0	Type I error, α (size)	Correct, power $1 - \beta$

Confidence interval, two-sided

$$\hat{\mu} \pm C_\alpha\,\hat{\sigma}/\sqrt{n}$$

Confidence interval, one-sided

$H_1 : \mu > \mu_0$: $[\hat{\mu} - C_\alpha\hat{\sigma}/\sqrt{n},\ \infty)$;

$H_1 : \mu < \mu_0$: $(-\infty,\ \hat{\mu} + C_\alpha\hat{\sigma}/\sqrt{n}]$

p-value

two-sided $2[1 - \Phi(|T|)]$; lower tail $\Phi(T)$;

upper tail $1 - \Phi(T)$

Difference in means, paired

$$T = \frac{\hat{\mu}_X - \hat{\mu}_Y}{\sqrt{(\hat{\sigma}_X^2 + \hat{\sigma}_Y^2 - 2\hat{\sigma}_{XY})/n}}$$

Difference in means, independent

$$T = \frac{\hat{\mu}_X - \hat{\mu}_Y}{\sqrt{\hat{\sigma}_X^2/n_X + \hat{\sigma}_Y^2/n_Y}}$$

QTA 7 Linear Regression

OLS estimates

$$\hat{\beta} = \frac{\sum(x_i - \bar{X})(y_i - \bar{Y})}{\sum(x_i - \bar{X})^2} = \frac{\hat{\sigma}_{XY}}{\hat{\sigma}_X^2} = \hat{\rho}_{XY}\frac{\hat{\sigma}_Y}{\hat{\sigma}_X};$$

$$\hat{\alpha} = \bar{Y} - \hat{\beta}\bar{X}$$

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Variance of the shocks

$$s^2 = \frac{1}{n-2}\sum \hat{\varepsilon}_i^2 = \frac{n}{n-2}\hat{\sigma}_Y^2(1-R^2)$$

Standard error of the slope

$$\text{s.e.}(\hat{\beta}) = \frac{s}{\sqrt{n}\,\hat{\sigma}_X}$$

t test, one coefficient

$$T = \frac{\hat{\beta}-\beta_0}{\text{s.e.}(\hat{\beta})}$$

same form for $\hat{\alpha}$

Confidence interval, coefficient

$$\hat{\beta} \pm C_\alpha\,\text{s.e.}(\hat{\beta})$$

C_α = two-sided critical value (1.96 at 95%)

Sums of squares

$$\text{ESS} = \sum(\hat{Y}_i - \bar{Y})^2;\quad \text{RSS} = \sum(Y_i - \hat{Y}_i)^2;$$
$$\text{TSS} = \sum(Y_i - \bar{Y})^2 = \text{ESS} + \text{RSS}$$

R²

$$R^2 = \frac{\text{ESS}}{\text{TSS}} = 1 - \frac{\text{RSS}}{\text{TSS}};\quad \text{one regressor: } R^2 = \hat{\rho}_{XY}^2$$

QTA 8 Regression with Multiple Explanatory Variables

R², multiple regressors

$$R^2 = \hat{\rho}^2(Y, \hat{Y})$$

Adjusted R²

$$\bar{R}^2 = 1 - \frac{\text{RSS}/(n-k-1)}{\text{TSS}/(n-1)} = 1 - \frac{n-1}{n-k-1}(1-R^2)$$

k excludes the constant

F test, q restrictions

$$F = \frac{(\text{RSS}_R - \text{RSS}_U)/q}{\text{RSS}_U/(n-k_U-1)} = \frac{(R_U^2 - R_R^2)/q}{(1-R_U^2)/(n-k_U-1)}$$

$F \sim F_{q,\,n-k_U-1}$

F statistic, all slopes zero

$$F = \frac{(\text{TSS}-\text{RSS})/k}{\text{RSS}/(n-k-1)} \sim F_{k,\,n-k-1}$$

F from t, uncorrelated regressors

$$F \approx \frac{T_1^2 + T_2^2}{2}$$

QTA 9 Regression Diagnostics

Omitted variable bias

$$\hat{\beta}_1 \to \beta_1 + \beta_2\delta,\ \delta = \frac{\text{Cov}(X_1, X_2)}{V(X_1)}$$

White test

nR^2 from regressing $\hat{\varepsilon}^2$ on the regressors, their squares and cross products $\sim \chi^2_{k(k+3)/2}$

Variance inflation factor

$$\text{VIF}_j = \frac{1}{1-R_j^2}$$

above 10 is excessive

Cook's distance

$$D_j = \frac{\sum_{i=1}^n (\hat{Y}_i^{(-j)} - \hat{Y}_i)^2}{k\,s^2}$$

$D_j > 1$ flags an outlier

QTA 10 Stationary Time Series

Autocorrelation

$$\rho_h = \gamma_h/\gamma_0$$

AR(1)

$$Y_t = \delta + \phi Y_{t-1} + \varepsilon_t,\ \varepsilon_t \sim WN(0,\sigma^2);$$
$|\phi| < 1$ stationary, $|\phi| = 1$ non-stationary

AR(1) moments

$$\mu = \frac{\delta}{1-\phi};\quad \gamma_0 = \frac{\sigma^2}{1-\phi^2};\quad \gamma(h) = \phi^{|h|}\gamma_0;\quad \rho(h) = \phi^{|h|}$$

μ is the mean-reverting level

AR(1) PACF

$\alpha(h) = \phi^{|h|}$ for $|h| \le 1$; 0 for $|h| \ge 2$

AR(p)

$$Y_t = \delta + \phi_1 Y_{t-1} + \cdots + \phi_p Y_{t-p} + \varepsilon_t$$

AR(p) moments

$$\mu = \frac{\delta}{1-\phi_1-\cdots-\phi_p};\quad \gamma_0 = \frac{\sigma^2}{1-\phi_1\rho_1-\cdots-\phi_p\rho_p}$$

AR(p) ACF and PACF

ACF decays, non-zero at all lags; PACF $\alpha(h) = 0$ for $h > p$

Characteristic equation

roots of $z^p - \phi_1 z^{p-1} - \cdots - \phi_p = 0$;
stationary if every root has $|z| < 1$
AR(2): $z^2 - \phi_1 z - \phi_2 = 0$

MA(1)

$Y_t = \mu + \theta\varepsilon_{t-1} + \varepsilon_t$, always covariance stationary;
$\theta > 0$ persistent, $\theta < 0$ mean-reverting

MA(1) moments

$$E[Y_t] = \mu;\quad \gamma_0 = (1+\theta^2)\sigma^2;$$
$$\rho(1) = \frac{\theta}{1+\theta^2},\ \rho(h) = 0 \text{ for } h \ge 2$$

MA(q)

$$Y_t = \mu + \varepsilon_t + \theta_1\varepsilon_{t-1} + \cdots + \theta_q\varepsilon_{t-q};\quad E[Y_t] = \mu;$$
$$\gamma_0 = \sigma^2(1 + \theta_1^2 + \cdots + \theta_q^2)$$

MA(q) ACF and PACF

$\rho(h) = 0$ for $h > q$; PACF decays, non-zero at all lags

ARMA(1,1)

$$Y_t = \delta + \phi Y_{t-1} + \theta\varepsilon_{t-1} + \varepsilon_t;\quad \mu = \frac{\delta}{1-\phi};$$
$$\gamma_0 = \frac{\sigma^2(1 + 2\phi\theta + \theta^2)}{1-\phi^2};\quad \gamma(h) = \phi\,\gamma(h-1) \text{ for } h \ge 2$$

> *Identification: the PACF of an AR(p) cuts off after lag p; the ACF of an MA(q) cuts off after lag q.*

Lag operator

$LY_t = Y_{t-1}$; $1 - a_1 L$ is invertible if $|a_1| < 1$

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Sample autocorrelation

$$\hat{\rho}_h = \hat{\gamma}_h / \hat{\gamma}_0$$

Box-Pierce

$$Q_{BP} = T \sum_{i=1}^{h} \hat{\rho}_i^2 \sim \chi_h^2$$

Ljung-Box

$$Q_{LB} = T(T+2) \sum_{i=1}^{h} \frac{\hat{\rho}_i^2}{T-i} \sim \chi_h^2$$

$Q_{BP} \approx Q_{LB}$ in large samples

Model selection

$$\hat{\sigma}^2 = \frac{1}{T} \sum_{t=1}^{T} \hat{\varepsilon}_t^2; \quad \text{AIC} = T \ln \hat{\sigma}^2 + 2k;$$
$$\text{BIC} = T \ln \hat{\sigma}^2 + k \ln T$$

Forecasts

AR(1): $E_T[Y_{T+h}] = \mu + \phi^h (Y_T - \mu) \to \mu$;

ARMA(1,1), one step: $E_T[Y_{T+1}] = \delta + \phi Y_T + \theta \varepsilon_T$

QTA 11 Non-Stationary Time Series

Linear trend

$$Y_t = \delta_0 + \delta_1 t + \varepsilon_t; \quad E[Y_t] = \delta_0 + \delta_1 t$$

Exponential and log-linear trend

$$Y_t = \beta_0 e^{\beta_1 t} \Leftrightarrow \ln Y_t = \ln \beta_0 + \beta_1 t$$

Seasonal dummies

$$Y_t = \delta + \gamma_1 I_{1t} + \cdots + \gamma_{s-1} I_{s-1,t} + \varepsilon_t$$

Random walk

$$Y_t = Y_0 + \sum_{i=1}^{t} \varepsilon_i; \quad V[Y_t] = t\sigma^2$$

$\Delta Y_t = Y_t - Y_{t-1}$ removes a unit root

ADF test

$$\Delta Y_t = \gamma Y_{t-1} + (\delta_0 + \delta_1 t) +$$
$$\lambda_1 \Delta Y_{t-1} + \cdots + \lambda_p \Delta Y_{t-p} + \varepsilon_t$$

statistic: t of $\hat{\gamma}$; $H_0 : \gamma = 0$ (unit root), $H_1 : \gamma < 0$ (stationary)

Forecasts

trend: $E_T[Y_{T+h}] = \delta_0 + \delta_1 (T+h)$;

seasonal: $E_T[Y_{T+h}] = \delta + \gamma_j, j = (T+h) \bmod s$

95% forecast interval

$$E_T[Y_{T+h}] \pm 1.96\,\sigma$$

QTA 12 Measuring Returns, Volatility and Correlation

Simple return

$$R_t = \frac{P_t - P_{t-1}}{P_{t-1}}; \quad 1 + R_T = \prod_{t=1}^{T} (1 + R_t)$$

Log return

$$r_t = \ln P_t - \ln P_{t-1}; \quad r_T = \sum_{t=1}^{T} r_t; \quad 1 + R_t = e^{r_t}$$

Jarque-Bera

$$JB = (T-1) \left[\frac{\hat{S}^2}{6} + \frac{(\hat{\kappa} - 3)^2}{24} \right] \sim \chi_2^2$$

5% critical value 5.99

Power law

$$P(X > x) = k x^{-\alpha} \Leftrightarrow \ln P(X > x) = \ln k - \alpha \ln x$$

Spearman rank correlation

$$\hat{\rho}_s = 1 - \frac{6 \sum d_i^2}{n(n^2 - 1)}$$

$d_i = \text{Rank}_{X_i} - \text{Rank}_{Y_i}$; distinct ranks

Kendall's τ

$$\hat{\tau} = \frac{n_c - n_d}{n(n-1)/2} = \frac{n_c}{n_c + n_d + n_t} - \frac{n_d}{n_c + n_d + n_t}$$

QTA 13 Simulation and Bootstrapping

Monte Carlo estimate

$$\hat{E}[g(X)] = \frac{1}{b} \sum_{i=1}^{b} g(X_i)$$

Sampling error

$$V\left[\hat{E}[g(X)]\right] = \frac{\sigma_g^2}{b}; \quad \text{standard error } \frac{\sigma_g}{\sqrt{b}}$$

Antithetic variables

$$\text{standard error} = \frac{\sigma_g \sqrt{1+\rho}}{\sqrt{b}}$$

smaller when $\rho < 0$

QTA 14 Machine-Learning Methods

Rescaling

$$\text{standardization } \tilde{x}_{ij} = \frac{x_{ij} - \mu_i}{\sigma_i};$$
$$\text{normalization } \tilde{x}_{ij} = \frac{x_{ij} - x_{i,\min}}{x_{i,\max} - x_{i,\min}}$$

Distances

$$\text{Euclidean } d_E = \sqrt{\sum_{i=1}^{m} (x_{iQ} - x_{iP})^2};$$
$$\text{Manhattan } d_M = \sum_{i=1}^{m} |x_{iQ} - x_{iP}|$$

K-means inertia

$$I = \sum_{j=1}^{n} d_j^2$$

Q-value update, Monte Carlo

$$Q_{\text{new}}(S, A) = Q_{\text{old}}(S, A) + \alpha [R - Q_{\text{old}}(S, A)]$$

R = total subsequent reward

Q-value update, temporal difference

$$Q_{\text{new}}(S, A) = Q_{\text{old}}(S, A) + \alpha [R + V(S') - Q_{\text{old}}(S, A)]$$

R = reward to the next decision;

$V(S')$ = highest current Q-value in the next state

QTA 15 Machine Learning and Prediction

Logistic regression

$$p = \frac{1}{1 + \exp[-(\alpha + \beta_1 x_1 + \cdots + \beta_k x_k)]};$$
$$\ln \frac{p}{1-p} = \alpha + \sum \beta_j x_j$$

Ridge and LASSO

ridge: $\min \text{RSS} + \lambda \sum \beta_j^2$; LASSO: $\min \text{RSS} + \lambda \sum |\beta_j|$

LASSO can set coefficients to exactly 0

Mean squared forecast error

$$\text{MSFE} = \frac{1}{n_{\text{test}}} \sum_{i=1}^{n_{\text{test}}} (y_i - \hat{y}_i)^2$$

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Confusion matrix	Formula
Accuracy	$\frac{TP+TN}{TP+TN+FP+FN}$
Precision	$\frac{TP}{TP+FP}$
Recall	$\frac{TP}{TP+FN}$
Error rate	$\frac{FP+FN}{TP+TN+FP+FN} = 1 - \text{accuracy}$

Decision tree impurity

$$\text{entropy} = -\sum_{i=1}^{M} p_i \log_2 p_i; \quad \text{Gini} = 1 - \sum_{i=1}^{M} p_i^2$$

FINANCIAL MARKETS AND PRODUCTS

FMP 2 Insurance Companies and Pension Plans

Survival probability

$$P(\text{survive to } N+1) = P(\text{survive to } N) \times (1 - q_N)$$

q_N = probability of death within the year from age N, given alive at N

Survival from age X to age Y

$$P(X \to Y) = \frac{P(\text{survive from birth to } Y)}{P(\text{survive from birth to } X)}$$

Break-even premium

PV of expected premiums = PV of expected payouts

each premium weighted by the probability of being alive to pay it;
each payout by the probability of death in that year

Ratio	Formula
Loss ratio	Payouts / premiums
Expense ratio	Expenses / premiums
Combined ratio	Loss ratio + expense ratio
Combined ratio after dividends	Combined ratio + dividends as % of premiums
Operating ratio	Combined ratio after dividends − investment income as % of premiums

FMP 3 Fund Management

Net asset value

$$\text{NAV} = \frac{\text{Fund assets} - \text{Fund liabilities}}{\text{Shares outstanding}}$$

Hedge fund return after fees

investor return = gross − management fee − incentive fee

incentive fee = incentive rate × max(0, return after management fee − hurdle rate), charged only on value above the high-water mark

FMP 4 Introduction to Derivatives

Forward payoff

long $S_T - K$; short $K - S_T$

option payoffs are in FMP 12

FMP 5 Exchanges and OTC Markets

Futures margin

variation margin = $\Delta F \times$ contract size $\times$ number of contracts

margin call when the balance falls below the maintenance margin; top up to the initial margin

Buying stock on margin

equity = market value − loan;

margin call when $\frac{\text{equity}}{\text{market value}} <$ maintenance ratio;

trigger price $P^* = P_0 \frac{1 - \text{IM}}{1 - \text{MM}}$

IM, MM = initial and maintenance margin ratios

FMP 8 Using Futures for Hedging

Basis

$$b_t = S_t - F_t$$

Hedged price

short hedge, price received $= F_0 + b_t$;

long hedge, price paid $= F_0 + b_t$

Minimum variance hedge ratio

$$h^* = \rho \frac{\sigma_S}{\sigma_F}; \quad \text{hedge effectiveness} = \rho^2$$

Optimal number of contracts

$$N^* = h^* \frac{Q_A}{Q_F}; \quad \text{tailing the hedge: } N^* = h^* \frac{V_A}{V_F} = h^* \frac{Q_A}{Q_F} \cdot \frac{S}{F}$$

Stock index hedge

$$N^* = \beta \frac{V_A}{V_F}$$

V_F = futures price × multiplier

Changing portfolio beta

$$N^* = (\beta^* - \beta)\frac{V_A}{V_F}$$

$N^* < 0$: short futures to lower beta;
$N^* > 0$: long futures to raise it

FMP 9 Foreign Exchange Markets

Real interest rate

$$R_{\text{real}} = \frac{1 + R_{\text{nom}}}{1 + R_{\text{infl}}} - 1 \approx R_{\text{nom}} - R_{\text{infl}}$$

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Bid-ask spread

$$\text{spread} = \text{ask} - \text{bid}; \quad \%\ \text{spread} = \frac{\text{ask} - \text{bid}}{\text{mid}}$$

Covered interest parity

$$F = S\left(\frac{1 + R_{YYY}}{1 + R_{XXX}}\right)^T$$

rate quoted XXXYYY (units of YYY per XXX)

Forward premium and points

$$\frac{F - S}{S} \approx (R_{YYY} - R_{XXX})\,T;$$

$$\text{forward points} = (F - S) \times 10{,}000$$

Uncovered interest parity

$$E(S_T) = S_0\left(\frac{1 + R_{YYY}}{1 + R_{XXX}}\right)^T$$

Purchasing power parity

$$\frac{S_T}{S_0} = \left(\frac{1 + \pi_{YYY}}{1 + \pi_{XXX}}\right)^T; \quad \%\ \text{change in } S \approx \pi_{YYY} - \pi_{XXX}$$

Appreciation and depreciation

XXXYYY moves from S_0 to S_1: XXX changes by $\dfrac{S_1 - S_0}{S_0}$;

YYY changes by $\dfrac{S_0 - S_1}{S_1}$

> *The two percentage changes differ, so work out each currency on its own base.*

FMP 10 Pricing Financial Forwards and Futures

Forward price	Annual compounding	Continuous
No income	$S_0(1+R)^T$	$S_0 e^{rT}$
Known income, PV I	$(S_0 - I)(1+R)^T$	$(S_0 - I)e^{rT}$
Known yield Q (index: dividend yield)	$S_0\left(\dfrac{1+R}{1+Q}\right)^T$	$S_0 e^{(r-q)T}$

Arbitrage

actual $F_0 >$ theoretical: buy the asset, short the forward;

actual $F_0 <$ theoretical: short the asset, go long the forward

Asset	Value of a long forward
No income	$S_0 - \dfrac{K}{(1+R)^T}$
Known income	$(S_0 - I) - \dfrac{K}{(1+R)^T}$
Known yield	$\dfrac{S_0}{(1+Q)^T} - \dfrac{K}{(1+R)^T}$
Any asset	$\dfrac{F_0 - K}{(1+R)^T}$; short $=-$ long

Short sale profit, dividend-paying stock

$$S_0 - S_T - \text{dividends paid while short}$$

plus interest on the proceeds if earned

FMP 11 Commodity Forwards and Futures

Forward with lease rate

$$F_0 = S_0\left(\frac{1+R}{1+l}\right)^T; \quad \text{implied } l = \left(\frac{S_0}{F_0}\right)^{1/T}(1+R) - 1$$

Storage cost and convenience yield

$$F_0 = (S_0 + U)\left(\frac{1+R}{1+Y}\right)^T;$$

$$\text{implied } Y = \left(\frac{S_0 + U}{F_0}\right)^{1/T}(1+R) - 1$$

U = PV of storage costs;

convenience yield belongs to consumption commodities

No-arbitrage upper bounds

$F_0 \le S_0(1+R)^T$; with storage: $F_0 \le (S_0 + U)(1+R)^T$

Continuous cost of carry

$$F_0 = S_0 e^{(c-y)T}$$

c = interest rate + storage cost rate, y = convenience yield

Expected future spot price

$$E(S_T) = F_0\frac{(1+X)^T}{(1+R)^T}$$

X = expected return on the commodity

FMP 12 Options Markets

Position	Payoff	Profit
Long call	$\max(S_T - K, 0)$	$\max(S_T - K, 0) - c_0$
Short call	$-\max(S_T - K, 0)$	$c_0 - \max(S_T - K, 0)$
Long put	$\max(K - S_T, 0)$	$\max(K - S_T, 0) - p_0$
Short put	$-\max(K - S_T, 0)$	$p_0 - \max(K - S_T, 0)$

FMP 13 Properties of Options

European call	Lower bound	Upper
No dividends	$\max(S_0 - \text{PV}(K), 0)$	S_0
Dividends	$\max(S_0 - \text{PV}(D) - \text{PV}(K), 0)$	S_0
Other assets	$\max(\text{PV}(F) - \text{PV}(K), 0)$	$\text{PV}(F)$

European put	Lower bound	Upper
No dividends	$\max(\text{PV}(K) - S_0, 0)$	$\text{PV}(K)$
Dividends	$\max(\text{PV}(D) + \text{PV}(K) - S_0, 0)$	$\text{PV}(K)$
Other assets	$\max(\text{PV}(K) - \text{PV}(F), 0)$	$\text{PV}(K)$

American call	Lower bound	Upper
No dividends	$\max(S_0 - \text{PV}(K), 0)$	S_0
Dividends	$\max(S_0 - K,\ S_0 - \text{PV}(D) - \text{PV}(K),\ 0)$	S_0

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American put	Lower bound	Upper
No dividends	$\max(K - S_0, 0)$	K
Dividends	$\max(K - S_0, \text{PV}(D) + \text{PV}(K) - S_0, 0)$	K

Put-call parity, European

non-dividend $c + \text{PV}(K) = p + S_0$;
dividend $c + \text{PV}(D) + \text{PV}(K) = p + S_0$;
forward form $c + \text{PV}(K) = p + \text{PV}(F)$

Put-call relation, American

non-dividend $S_0 - K \le C - P \le S_0 - \text{PV}(K)$;
dividend $S_0 - \text{PV}(D) - K \le C - P \le S_0 - \text{PV}(K)$

Early exercise of an American call

exercise just before an ex-dividend date can be optimal only if $D > K - K^*$;

never exercise early without dividends

K^* = K discounted from the next ex-dividend date (or maturity) back to this one

Increase in	European call	European put	American call	American put
S_0	+	–	+	–
K	–	+	–	+
T	?	?	+	+
σ	+	+	+	+
r	+	–	+	–
Dividends	–	+	–	+

FMP 14 Trading Strategies

Option and stock equivalences

protective put $S + P = C + \text{PV}(K)$;
covered call $S - C = \text{PV}(K) - P$

Payoff at T	$S_T \le K_1$	$K_1 < S_T \le K_2$	$S_T > K_2$
Bull: long call K_1, short call K_2	0	$S_T - K_1$	$K_2 - K_1$
Bear: long put K_2, short put K_1	$K_2 - K_1$	$K_2 - S_T$	0
Box: bull call + bear put	$K_2 - K_1$	$K_2 - K_1$	$K_2 - K_1$
Strangle: long put K_1, long call K_2	$K_1 - S_T$	0	$S_T - K_2$

Box spread value

$$\text{PV}(K_2 - K_1)$$

Butterfly (calls)

0 below K_1; $S_T - K_1$ from K_1 to K_2;
$K_3 - S_T$ from K_2 to K_3; 0 above K_3

long call K_1, short 2 calls K_2, long call K_3; $K_2 = (K_1 + K_3)/2$

Straddle

long call + long put, same K: payoff $|S_T - K|$

Strip and strap

strip = 1 call + 2 puts (bearish tilt);
strap = 2 calls + 1 put (bullish tilt)
same strike and maturity

Principal protected note

zero-coupon bond paying L + call option;
feasible if $L - Le^{-rT} \ge$ option cost

FMP 15 Exotic Options

Zero-cost call

payoff $\max(S_T - K - A, -A)$, $A = c(1 + R)^T$

Gap options

strike K_1, trigger K_2: call pays $S_T - K_1$ if $S_T > K_2$;
put pays $K_1 - S_T$ if $S_T < K_2$; else 0
the payoff can be negative

Payoff	Call	Put
Average price	$\max(S_{\text{avg}} - K, 0)$	$\max(K - S_{\text{avg}}, 0)$
Average strike	$\max(S_T - S_{\text{avg}}, 0)$	$\max(S_{\text{avg}} - S_T, 0)$
Floating lookback	$S_T - S_{\min}$	$S_{\max} - S_T$
Fixed lookback	$\max(S_{\max} - K, 0)$	$\max(K - S_{\min}, 0)$
Cash-or-nothing	Q if $S_T > K$	Q if $S_T < K$
Asset-or-nothing	S_T if $S_T > K$	S_T if $S_T < K$

Barrier parity

knock-in + knock-out = vanilla option
same barrier, strike and maturity

Exchange option

payoff $\max(U_T - V_T, 0)$
receive asset U, give up asset V

Volatility and variance swaps

receive realized: $L_{\text{vol}}(\sigma - \sigma_K)$; $L_{\text{var}}(\sigma^2 - V_K)$
$V_K = \sigma_K^2$; $L_{\text{var}} = L_{\text{vol}}/(2\sigma_K)$; paying realized reverses the signs

FMP 16 Properties of Interest Rates

Compounding

$FV = PV(1 + R/m)^{mT}$; $PV = FV(1 + R/m)^{-mT}$;
continuous: $FV = PVe^{RT}$, $PV = FVe^{-RT}$

Converting rates

$$R_2 = m_2\left[\left(1 + \frac{R_1}{m_1}\right)^{m_1/m_2} - 1\right]; \quad R_m = m\left(e^{R_c/m} - 1\right);$$
$$R_c = m\ln(1 + R_m/m)$$

Bond price from yield

$$B = \frac{c}{2}\sum_{i=1}^{2T}\left(1 + \frac{y}{2}\right)^{-i} + 100\left(1 + \frac{y}{2}\right)^{-2T}$$

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Bond price from spot rates

$$B = \sum c_i e^{-R_i t_i}, \text{ or } \sum \frac{c_i}{(1 + R_i/m)^{m t_i}}$$

Par yield

$$c = \frac{(100 - 100\, d_T)\, m}{A}$$

d_T = final discount factor;

A = sum of coupon-date discount factors

Duration

$$\text{Macaulay } D = \sum t_i \frac{PV_i}{B}; \quad \text{modified } D_{\text{mod}} = \frac{D_{\text{Mac}}}{1 + y/m};$$

$$\text{dollar } D_{\$} = D_{\text{mod}}\, B$$

Convexity

$$C = \sum t_i^2 \frac{PV_i}{B}$$

Price change

$$\Delta B = -DB\, \Delta y + \tfrac{1}{2} CB (\Delta y)^2;$$

$$\text{with dollar duration: } \Delta B = -D_{\$}\, \Delta y$$

Forward rates

$$\text{discrete } F = \left[\frac{(1 + R_2)^{T_2}}{(1 + R_1)^{T_1}}\right]^{1/(T_2 - T_1)} - 1;$$

$$\text{continuous } F = \frac{R_2 T_2 - R_1 T_1}{T_2 - T_1}$$

FRA, pay fixed

$$\text{payoff at the start of the period } \frac{(R - R_K)\, \tau L}{1 + R\tau};$$

value today $= (R_F - R_K)\, \tau L$ discounted from the period end

receive fixed: signs reversed

FMP 17 Corporate Bonds

Expected loss and return

$$\text{expected loss rate} = \text{PD} \times (1 - \text{RR});$$

$$\text{expected return} = r_f + \text{credit spread} - \text{expected loss rate}$$

Default rates

$$\text{issue default rate} = \frac{\text{defaulted issues}}{\text{total issues}};$$

$$\text{dollar default rate} = \frac{\text{par value defaulted}}{\text{total par value}}$$

FMP 18 Mortgages and Mortgage-Backed Securities

Monthly mortgage payment

$$X = \frac{A\,(R/12)}{1 - (1 + R/12)^{-12T}}$$

A = amount borrowed

Interest and principal

$$\text{Interest}_t = \text{Balance}_{t-1} \times R/12;$$

$$\text{Principal}_t = X - \text{Interest}_t;$$

$$\text{Balance}_t = \text{Balance}_{t-1} - \text{Principal}_t$$

WAC and WAM

$$\text{WAC} = \sum w_i c_i; \quad \text{WAM} = \sum w_i L_i; \quad w_i = \frac{P_i}{\sum P_i}$$

P_i = remaining principal of mortgage i

SMM and CPR

$$\text{SMM} = \frac{\text{prepayment in the month}}{\text{beginning balance} - \text{scheduled principal}};$$

$$\text{CPR} = 1 - (1 - \text{SMM})^{12}$$

Dollar roll value

$$A - B + C - D$$

A = sale price incl. AI; B = repurchase price incl. AI;

C = interest on proceeds for the month;

D = coupon and principal given up

Refinancing incentive

$$I = \text{WAC} - R$$

R = current mortgage rate

FMP 19 Interest Rate Futures

Market	Day count
Treasury bonds (and AI on futures delivery)	Actual/actual
Corporate and municipal bonds	30/360
T-bills and money market	Actual/360

Accrued interest and dirty price

$$\text{AI} = \frac{\text{days since last coupon}}{\text{days in coupon period}} \times \text{coupon};$$

$$\text{dirty} = \text{clean} + \text{AI}$$

T-bill

$$Q = \frac{360}{n}(100 - C); \quad C = 100 - \frac{Qn}{360}$$

Q = quote, C = cash price per 100, n = days to maturity

Conversion factor

price per 1 of face of the delivered bond at a 6% yield (semiannual), maturity rounded down to the nearest 3 months

Delivery

$$\text{cash received} = \text{settlement price} \times \text{CF} + \text{AI};$$

$$\text{CTD minimizes quoted price} - \text{settlement price} \times \text{CF}$$

Treasury futures price

$$F_0 = (S_0 - I)(1 + R)^T, \text{ or } (S_0 - I)e^{rT}$$

I = PV of coupons during the futures life

Duration-based hedge

$$N^* = \frac{P\, D_P}{V_F\, D_F}$$

P, D_P at hedge maturity; V_F = contract price

DV01 hedge

$$N = -\frac{E_V}{E_F}$$

E = value gain for a 1 bp fall in all rates; $N > 0$ long, $N < 0$ short

FMP 20 Swaps

Comparative advantage

$$\text{total gain} = |\Delta_{\text{fixed}} - \Delta_{\text{floating}}|$$

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Swap position	Value
Pay fixed, receive floating	$B_{\text{float}} - B_{\text{fix}}$
Receive fixed, pay floating	$B_{\text{fix}} - B_{\text{float}}$
Currency: receive domestic (S_0 = domestic per foreign)	$B_D - S_0 B_F$
Currency: receive foreign	$S_0 B_F - B_D$

Bond values inside a swap

$$B_{\text{fix}} = \sum c\, \exp(-r_i t_i) + L \exp(-r_n t_n);$$
$$B_{\text{float}} = (L + k^*) \exp(-r^* t^*)$$

$B_{\text{float}} = L$ immediately after a reset

Swap value from FRAs

pay fixed: $V = \sum \text{PV}\left[(R_{F,i} - R_K)\,\tau L\right]$

$R_{F,i}$ = forward rate for period i

Currency swap from forwards

receive foreign: $V = \sum \text{PV}(\text{FCF}_i \times F_i - \text{DCF}_i)$

FCF, DCF = foreign and domestic cash flows;
F_i = forward rate (domestic per foreign);
receive domestic: sign reversed

VALUATION AND RISK MODELS

VRM 1 Measures of Financial Risk

Two-asset portfolio

$$\mu_P = w_1\mu_1 + w_2\mu_2;$$
$$\sigma_P = \sqrt{w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2\rho w_1 w_2 \sigma_1 \sigma_2}$$

Normal distribution of	VaR at confidence X	ES at confidence X
Losses (μ_L, σ_L)	$\mu_L + \sigma_L z$	$\mu_L + \sigma_L \dfrac{e^{-z^2/2}}{(1-X)\sqrt{2\pi}}$
Returns (μ_P, σ_P)	$\sigma_P z - \mu_P$	$\sigma_P \dfrac{e^{-z^2/2}}{(1-X)\sqrt{2\pi}} - \mu_P$

VaR from a loss distribution adds the mean; from a return distribution it subtracts the mean.

VRM 2 Calculating and Applying VaR

Linear portfolio

$$\Delta P = \sum q_i\, \Delta r_i$$

$q_i = n_i S_i$ (amount invested), $\Delta r_i = \Delta S_i / S_i$

Risk factor sensitivities

linear $\Delta P = \sum \delta_i\, \Delta S_i;$

delta-gamma $\Delta P = \sum \delta_i\, \Delta S_i + \frac{1}{2}\sum \gamma_i (\Delta S_i)^2$

Zero-mean VaR and ES

$$\text{VaR} = \sigma_P z; \quad \text{ES} = \sigma_P \frac{e^{-z^2/2}}{(1-X)\sqrt{2\pi}}$$

Square-root-of-time rule

$$\text{VaR}(T, X) = \sqrt{T}\,\text{VaR}(1, X);$$
$$\text{ES}(T, X) = \sqrt{T}\,\text{ES}(1, X)$$

needs iid returns and a zero mean

Delta-normal VaR of an option

$$\text{VaR} = |\delta| \times S \times \sigma \times z$$

Historical simulation

VaR = the $n(1-X)$-th worst of the n scenario losses;
ES = average of the $n(1-X)$ worst losses

VRM 3 Measuring and Monitoring Volatility

Variance from daily returns

$$\sigma_n^2 = \frac{1}{m}\sum_{i=1}^{m} u_{n-i}^2$$

$u_i = (S_i - S_{i-1})/S_{i-1}$; mean return taken as 0

EWMA

$$\sigma_n^2 = \lambda\,\sigma_{n-1}^2 + (1-\lambda)\,u_{n-1}^2$$

RiskMetrics $\lambda = 0.94$

GARCH(1,1)

$$\sigma_n^2 = \omega + \alpha u_{n-1}^2 + \beta\sigma_{n-1}^2$$

$\omega = \gamma V_L, \gamma = 1 - \alpha - \beta$

Long-run variance

$$V_L = \frac{\omega}{1-\alpha-\beta}$$

GARCH variance forecast

$$E[\sigma_{n+t}^2] = V_L + (\alpha+\beta)^t(\sigma_n^2 - V_L)$$

EWMA covariance and correlation

$$\text{cov}_n = \lambda\,\text{cov}_{n-1} + (1-\lambda)\,x_{n-1}y_{n-1}; \quad \rho_n = \frac{\text{cov}_n}{\sigma_{x,n}\,\sigma_{y,n}}$$

x, y = returns on day $n-1$

VRM 4 External and Internal Credit Ratings

Hazard rate

$$\text{PD}(0,t) = 1 - e^{-\bar{h}t}; \quad \text{survival } S(t) = e^{-\bar{h}t}$$

Average hazard rate from survival

$$\bar{h} = -\frac{\ln S(t)}{t}$$

Default between t_1 and t_2

$$\exp(-\bar{h}_1 t_1) - \exp(-\bar{h}_2 t_2)$$

$\bar{h}_2$ = average hazard rate to t_2

Conditional PD

$$P(\text{default in year } t \mid \text{survived to } t-1) = \frac{S(t-1) - S(t)}{S(t-1)}$$

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Expected loss

$$\text{EL} = \text{PD} \times \text{LGD} \times \text{EAD}$$

$\text{LGD} = 1 - \text{RR}$; in % terms $\text{EL} = \text{PD} \times \text{LGD}$

Rating transition matrix

$$n\text{-year matrix} = M^n$$

assumes independence across years;
cumulative PD is the default column

VRM 6 Measuring Credit Risk

Loss on one loan

mean $p_i L_i(1 - R_i)$; SD $\sigma_i = \sqrt{p_i - p_i^2}\, L_i(1 - R_i)$

Portfolio of identical loans

$$\sigma_P^2 = n\sigma^2 + n(n-1)\rho\sigma^2; \quad \frac{\sigma_P}{nL} = \frac{\sigma\sqrt{1 + (n-1)\rho}}{L\sqrt{n}}$$

One-factor Gaussian copula

$$U_i = a_i F + \sqrt{1 - a_i^2}\, Z_i; \quad \rho(U_i, U_j) = a_i a_j$$

Vasicek WCDR, 99.9%

$$\text{WCDR} = N\left(\frac{N^{-1}(\text{PD}) - \sqrt{\rho}\, N^{-1}(0.001)}{\sqrt{1-\rho}}\right)$$

Credit risk capital

worst-case loss $= \text{WCDR} \times \text{LGD} \times \text{EAD}$;
capital $= \text{UL} = (\text{WCDR} - \text{PD}) \times \text{LGD} \times \text{EAD}$;
loan by loan: $\sum(\text{WCDR}_i - \text{PD}_i)\,\text{LGD}_i\,\text{EAD}_i$

Euler contribution

$$\text{Contribution}_i = x_i \frac{\partial F}{\partial x_i}; \quad \sum_i \text{Contribution}_i = F$$

F = risk measure, x_i = size of loan i

VRM 7 Operational Risk

Basic indicator approach

capital = 15% × average annual gross income over 3 years

gross income = interest earned − interest paid + non-interest income

Standardized approach

capital = 3-year average of $\sum \beta_k\, \text{GI}_k$

β = 12% retail banking, asset management, retail brokerage;
15% commercial banking, agency services;
18% corporate finance, trading and sales, payment and settlement

SMA loss component

$$7X + 7Y + 5Z$$

X = average annual op loss; Y = from losses above €10m;
Z = from losses above €100m (last 10 years)

Loss frequency (Poisson)

$$P(n) = \frac{e^{-\lambda}\lambda^n}{n!}$$

Scaling external loss data

$$\text{Loss}_A = \text{Loss}_B \left(\frac{\text{Revenue}_A}{\text{Revenue}_B}\right)^{\beta}$$

β = scaling exponent estimated from data

Power law

$$P(v > x) = Kx^{-\alpha}$$

VRM 9 Pricing Conventions, Discounting and Arbitrage

Present value

$$PV = \sum CF_t\, d_t$$

d_t = discount factor for time t

Money-market yield from a discount quote

$$\frac{Qn/360}{100 - Qn/360}$$

360 to 365 day basis

$$r_{365} = r_{360} \times \frac{365}{360}$$

VRM 10 Interest Rates

Discount factors

semiannual $d_t = (1 + z_t/2)^{-2t}$; continuous $d_t = e^{-z_t t}$

Bond value

$$V = \frac{c}{2}A_T + 100\, d_T; \quad V = 100 + \frac{c - p}{2}A_T$$

A_T = sum of discount factors; p = par rate

Six-month forward rate

$$1 + \frac{f}{2} = \frac{(1 + z_{T+0.5}/2)^{2T+1}}{(1 + z_T/2)^{2T}}$$

Spot rate from forwards

$$(1 + z_n/2)^n = (1 + f_1/2)(1 + f_2/2)\cdots(1 + f_n/2)$$

n = number of half-years

VRM 11 Bond Yields and Return Calculations

Annuity and perpetuity

annuity $\frac{c}{y}\left[1 - (1 + y/2)^{-2T}\right]$; perpetuity $\frac{c}{y}$

c = annual coupon, paid semiannually

Realized return

$$\text{gross } R = \frac{P_{\text{end}} + \text{coupons} - P_{\text{begin}}}{P_{\text{begin}}};$$

$$\text{net} = R - \frac{\text{financing cost}}{P_{\text{begin}}};$$

several periods: $(1 + R_1)\cdots(1 + R_n) - 1$

Spread

$$P = \sum_t \frac{CF_t}{[1 + (z_t + s)/2]^{2t}}, \text{ solve for } s$$

P&L decomposition

P&L = carry roll-down + rate changes + spread changes

carry roll-down = price change from passing time plus coupon, on the assumed curve (forwards realized, or curve unchanged)

VRM 12 Applying Duration, Convexity and DV01

DV01

$$\text{DV01} = -\frac{\Delta P}{10{,}000 \times \Delta r}$$

Δr in decimal

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Effective duration

$$D = \frac{P_- - P_+}{2P\,\Delta r}; \quad \text{one-sided: } D = -\frac{1}{P}\frac{\Delta P}{\Delta r}$$

Effective convexity

$$C = \frac{P_+ + P_- - 2P}{P\,(\Delta r)^2}$$

Price change

$$\Delta P = -DP\,\Delta r + \tfrac{1}{2}CP\,(\Delta r)^2$$

Portfolio measures

$$\mathrm{DV01}_P = \sum \mathrm{DV01}_i; \quad D_P = \sum \frac{V_i}{V} D_i; \quad C_P = \sum \frac{V_i}{V} C_i$$

Hedge face amount

$$F_H = -F_P \frac{\mathrm{DV01}_P}{\mathrm{DV01}_H}$$

DV01s per 100 face

Duration and convexity hedge

$$\text{one bond: } P_H = -\frac{V D_V}{D_H};$$

$$\text{two bonds: } P_1 D_1 + P_2 D_2 = -V D_V;$$

$$P_1 C_1 + P_2 C_2 = -V C_V$$

Barbell matching a bullet

$$V_S + V_L = V_{\text{bullet}} \text{ and } V_S D_S + V_L D_L = V_{\text{bullet}} D_{\text{bullet}}$$

the barbell has higher convexity

VRM 13 Non-Parallel Term Structure Shifts and Hedging

Key rate and bucket 01s

$\mathrm{DV01} = \sum_i \mathrm{KR01}_i$; also $\mathrm{DV01} = \sum$ forward bucket 01s

Portfolio SD from key rates

$$\sigma_P = \sqrt{\sum_i \sum_j \rho_{ij}\sigma_i\sigma_j\,\mathrm{KR01}_i\,\mathrm{KR01}_j}$$

σ_i = SD of daily change in key rate i (bp);
ρ_{ij} = correlation of those changes

Portfolio SD from principal components

$$\sigma_P = \sqrt{\sigma_1^2 f_1^2 + \sigma_2^2 f_2^2 + \sigma_3^2 f_3^2}$$

f_i = value change per unit of factor i

Duration from a 01

$$\text{Duration} = \frac{10{,}000 \times 01 \text{ measure}}{\text{portfolio value}}$$

VRM 14 Binomial Trees

One step

$$f = e^{-rT}[pf_u + (1-p)f_d]; \quad p = \frac{e^{rT} - d}{u - d};$$

$$u = e^{\sigma\sqrt{T}},\ d = 1/u$$

$S_u = S_0 u,\ S_d = S_0 d$

Several steps

$$f = e^{-r\Delta t}[pf_u + (1-p)f_d]; \quad u = e^{\sigma\sqrt{\Delta t}},\ d = 1/u$$

American option node

$$f = \max\left(\text{exercise value},\ e^{-r\Delta t}[pf_u + (1-p)f_d]\right)$$

Delta

$$\Delta = \frac{f_u - f_d}{S_0 u - S_0 d}$$

Underlying	a in $p = (a-d)/(u-d)$
Non-dividend stock	$e^{r\Delta t}$
Dividend yield q or index	$e^{(r-q)\Delta t}$
Currency	$e^{(r-r_f)\Delta t}$
Futures	1

VRM 15 The Black-Scholes-Merton Model

Lognormal stock price

$$\frac{\Delta S}{S} \sim N(\mu\Delta t,\ \sigma^2\Delta t); \quad E(S_T) = S_0 e^{\mu T}$$

Distribution of $\ln S_T$

$$E(\ln S_T) = \ln S_0 + (\mu - \sigma^2/2)T; \quad \text{SD} = \sigma\sqrt{T}$$

Realized return

$$R = \frac{1}{T}\ln\frac{S_T}{S_0} \sim N\left(\mu - \frac{\sigma^2}{2},\ \frac{\sigma^2}{T}\right)$$

Historical volatility

$$u_i = \ln(S_i/S_{i-1}); \quad \hat{\sigma} = \frac{s}{\sqrt{\tau}}$$

s = sample SD of u_i; τ = interval length in years

BSM, no dividends

$$c = S_0 N(d_1) - Ke^{-rT}N(d_2);$$

$$p = Ke^{-rT}N(-d_2) - S_0 N(-d_1)$$

d_1 and d_2

$$d_1 = \frac{\ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}; \quad d_2 = d_1 - \sigma\sqrt{T}$$

Dividend yield q

$$c = S_0 e^{-qT}N(d_1) - Ke^{-rT}N(d_2);$$

$$p = Ke^{-rT}N(-d_2) - S_0 e^{-qT}N(-d_1);$$

$$d_1 = \frac{\ln(S_0/K) + (r - q + \sigma^2/2)T}{\sigma\sqrt{T}}$$

same for a stock index

Currency options

$$q = r_f\text{: } c = S_0 e^{-r_f T}N(d_1) - Ke^{-rT}N(d_2);$$

$$p = Ke^{-rT}N(-d_2) - S_0 e^{-r_f T}N(-d_1)$$

Futures options (Black)

$$c = e^{-rT}[F_0 N(d_1) - KN(d_2)];$$

$$p = e^{-rT}[KN(-d_2) - F_0 N(-d_1)];$$

$$d_1 = \frac{\ln(F_0/K) + \sigma^2 T/2}{\sigma\sqrt{T}}$$

Discrete dividends

$$c = [S_0 - \mathrm{PV}(D)]N(d_1) - Ke^{-rT}N(d_2)$$

replace S_0 by $S_0 - \mathrm{PV}(D)$ in every no-dividend formula, including d_1

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Warrants

$$\text{value per warrant} = \frac{N}{N+M} \times c;$$

$$\text{dilution cost to existing shareholders} = M \times \text{value per warrant}$$

c = BSM call on the stock; N = shares outstanding; M = warrants issued

VRM 16 The Greeks

Greek, no income	Call	Put
Delta	$N(d_1)$	$N(d_1) - 1$
Gamma	$\frac{N'(d_1)}{S_0\sigma\sqrt{T}}$	same as call
Vega	$S_0\sqrt{T}\,N'(d_1)$	same as call
Rho	$KTe^{-rT}N(d_2)$	$-KTe^{-rT}N(-d_2)$

Theta, no income

$$\text{call } -\frac{S_0N'(d_1)\sigma}{2\sqrt{T}} - rKe^{-rT}N(d_2);$$

$$\text{put } -\frac{S_0N'(d_1)\sigma}{2\sqrt{T}} + rKe^{-rT}N(-d_2)$$

Standard normal density

$$N'(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$$

Greek, yield q	Call	Put
Delta	$e^{-qT}N(d_1)$	$e^{-qT}[N(d_1) - 1]$
Gamma	$\frac{N'(d_1)\,e^{-qT}}{S_0\sigma\sqrt{T}}$	same as call
Vega	$S_0\sqrt{T}\,N'(d_1)\,e^{-qT}$	same as call
Rho	$KTe^{-rT}N(d_2)$	$-KTe^{-rT}N(-d_2)$

Theta, yield q, call

$$-\frac{S_0N'(d_1)\sigma e^{-qT}}{2\sqrt{T}} + qS_0N(d_1)e^{-qT} - rKe^{-rT}N(d_2)$$

Theta, yield q, put

$$-\frac{S_0N'(d_1)\sigma e^{-qT}}{2\sqrt{T}} - qS_0N(-d_1)e^{-qT} + rKe^{-rT}N(-d_2)$$

d_1 uses $r - q$, as in VRM 15

Currency and futures options

use $q = r_f$ for currencies; $q = r$ for futures

Long position	Delta	Gamma	Vega
Call	+	+	+
Put	–	+	+
Stock	1	0	0
Forward, non-dividend stock	1	0	0
Futures, non-dividend stock	e^{rT}	0	0
Forward, yield q	e^{-qT}	0	0
Futures, yield q	$e^{(r-q)T}$	0	0

Short positions reverse every sign in the table above.

Portfolio Greeks

$$\Delta_P = \sum w_i\Delta_i; \quad \Gamma_P = \sum w_i\Gamma_i; \quad \nu_P = \sum w_i\nu_i$$

w_i = number of options of type i

Gamma-neutral, then delta-neutral

$$\text{options to trade } n = -\frac{\Gamma_P}{\Gamma_T}; \quad \text{then shares} = -(\Delta_P + n\Delta_T)$$

Theta, delta and gamma

$$\Theta + rS\Delta + \tfrac{1}{2}\sigma^2S^2\Gamma = r\Pi;$$

$$\text{delta-neutral: } \Theta + \tfrac{1}{2}\sigma^2S^2\Gamma = r\Pi$$

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